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# The connection between factorization properties and closed-form solutions of certain linear dyadic differential operators

Werner S Weiglhofer

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, UK

E-mail: wsw@maths.gla.ac.uk

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**Abstract.** Representation of the electromagnetic field in terms of dyadic Green functions leads to the requirement to solve dyadic partial differential equations for the dyadic Green functions. While a formal solution for the infinite-medium problem can be given in a straightforward manner for even the most general, linear medium, the extraction of closed-form expressions is a complicated issue. The existence of such expressions for the dyadic Green functions is closely linked to the factorization properties of the determinant operator, which is associated with the dyadic differential operator of the dyadic Green functions. This connection is investigated for a special type of homogeneous, anisotropic dielectric medium.

## 1. Introduction

One of the standard solution methods of any linear field theory is the representation of the fields—which may have scalar, vector or tensorial character—in terms of Green functions. In classical electromagnetic theory, this mapping from the vector sources to the vector electromagnetic field is facilitated by dyadic Green functions (sometimes also called Green tensors). Consequently, the vector partial differential equations for the electromagnetic field vectors are replaced by dyadic differential equations for the dyadic Green functions.

For the purpose of this communication we will consider the equation†

$$\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}}) \cdot \underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}')\underline{\underline{I}}. \tag{1}$$

Therein, $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ is the dyadic Green function (specifically, of the electric type) pertaining to frequency-dependent electromagnetic field phasors with a harmonic time dependence of $\exp(-\mathrm{i}\omega t)$. The implicit dependence on $\omega$ is suppressed henceforth. The Dirac delta function is denoted by $\delta(\boldsymbol{x} - \boldsymbol{x}')$ and $\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}})$ is a dyadic differential operator of second order given by

$$\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}}) = \nabla \times \nabla \times \underline{\underline{I}} - \omega^2 \mu \underline{\underline{\epsilon}}. \tag{2}$$

The specific form of (2) already incorporates the constitutive characterization of the medium in which the electromagnetic process takes place. In this instance it is a homogeneous, *anisotropic*

† The notation in this paper is such that vectors are in bold face whereas dyadics are in normal face and underlined twice. Contraction of indices is symbolized by a dot; that is, $\boldsymbol{a} \cdot \boldsymbol{b}$ represents $\sum_i a_i b_i$, whereas $\underline{\underline{A}} = \boldsymbol{ab}$ is a dyadic with elements $A_{ij} = a_i b_j$. The vectors to observation and source points are $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively; the unit dyadic is $\underline{\underline{I}}$. The superscript $^{-1}$ indicates inversion of dyadics and differential operators.

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*dielectric* medium: the anisotropy is contained in a permittivity dyadic $\underline{\underline{\epsilon}}$, whereas its magnetic isotropy is described by a scalar permeability $\mu$.

Radiation and scattering problems depend crucially on the availability of a solution of (1). In particular, it is the infinite-medium solution that is of greatest interest and especially in closed form. The term *closed form* indicates that $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ should be expressible through simple mathematical functions (which will most often be *scalar* Green functions of second-order Helmholtz-like operators) and derivatives and linear combinations thereof. It does not include representations in terms of integrals—as (1) is linear, such representations can always be achieved with spatial Fourier transforms.

The task of finding closed-form solutions of (1) becomes increasingly complicated the more complex the medium is that is described by $\underline{\underline{\epsilon}}$. The derivation of the infinite-medium solution of $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ for an *isotropic* medium, where $\underline{\underline{\epsilon}} = \epsilon \underline{\underline{I}}$ and $\epsilon$ is a scalar parameter, is a standard textbook example (see, e.g., [1]). When anisotropy is considered, the mathematical analysis becomes considerably more involved. The simplest type of anisotropy is *uniaxiality*, expressed by

$$\underline{\underline{\epsilon}}_{\text{uni}} = \epsilon_a \underline{\underline{I}} + \epsilon_b \boldsymbol{u}\boldsymbol{u} \tag{3}$$

where $\epsilon_a$, $\epsilon_b$ are two scalars and $\boldsymbol{u}$ is a unit vector. Closed-form expressions for $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ for such a medium were first given by Chen [2]. Other mediums have yielded closed-form solutions (also in the realm of *bianisotropy*, which adds magnetoelectric coupling to the simpler anisotropic problem considered here) and the interested reader is referred to two reviews for specific formulae [3, 4].

Further generalization leads to *biaxiality* [2, 5] with

$$\underline{\underline{\epsilon}}_{\text{bi}} = \epsilon_a \underline{\underline{I}} + \frac{\epsilon_b}{2} (\boldsymbol{u}_m \, \boldsymbol{u}_n + \boldsymbol{u}_n \, \boldsymbol{u}_m) \tag{4}$$

where $\boldsymbol{u}_m$ and $\boldsymbol{u}_n$ are two unit vectors that are in general not parallel or anti-parallel (if they are, (4) reduces to the uniaxial (3)). A closed-form expression for the infinite-medium dyadic Green function of a biaxial dielectric medium has been elusive to date despite significant efforts. Only expressions in terms of reduced integral representations or asymptotic/numerical formulae have been obtained [6–9].

The failure to derive a closed-form, infinite-medium dyadic Green function for a biaxial medium is not surprising in view of some results pertaining to the so-called determinant operator of $\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}})$. An important step towards a closed-form solution is that the determinant operator (see (8) for its definition), which is always of fourth order—a fact determined by the very structure of the Maxwell equations—can be factorized into a product of two second-order operators. The topic of factorization was investigated before and it was first stated in [10] (see also [11–13] for extensions to bianisotropic mediums) that a factorization of the determinant operator can be achieved if the dyadic permittivity $\underline{\underline{\epsilon}}$ has the structure†

$$\underline{\underline{\epsilon}}_{\text{fact}} = \lambda \underline{\underline{I}} + \boldsymbol{a}\boldsymbol{b} \tag{5}$$

where $\lambda$ is a scalar and $\boldsymbol{a}$, $\boldsymbol{b}$ are two vectors.

It must first be observed that the form (5) is only *sufficient* and not necessary for factorization. Furthermore, the relation between factorization and the availability of closed-form solutions is not clear. In fact, some years ago investigations of certain types of *uniaxial bianisotropic mediums* provided an example where closed-form solutions for dyadic Green functions could not be obtained despite factorization, first observed in [14] and explored further in [15, 16].

† Equation (5) is a specialization of formulae given in [10, 11], where a more general anisotropic medium was considered that also included a dyadic permeability.

Therefore, the motivation for this paper lies in gaining a more thorough understanding of the connection between factorization and closed-form solutions within the context of anisotropic dielectric mediums. This will be done by investigating the dyadic Green function of a medium with a special form of anisotropy.

## 2. An anisotropic dielectric medium

We shall thus consider an anisotropic dielectric medium described by

$$\underline{\underline{\epsilon}} = \epsilon_a \underline{\underline{I}} + \epsilon_b \boldsymbol{u}_m\, \boldsymbol{u}_n \tag{6}$$

where $\epsilon_a$ and $\epsilon_b$ are once again scalars and $\boldsymbol{u}_m$ and $\boldsymbol{u}_n$ are distinct unit vectors. The form (6) is equivalent to (5) as can be seen without difficulty. Such a medium is not *reciprocal* as $\underline{\underline{\epsilon}}$ is not equal to its transposed dyadic [17].

The uniaxial medium defined in (3) appears as a specialization of (6) when $\boldsymbol{u}_m = \boldsymbol{u}_n$. It is mentioned parenthetically that application of this specialization at any stage in this manuscript leads back to the correct expressions pertaining to a uniaxial dielectric medium. However, (6) also has a connection to a biaxial structure. Upon decomposition into symmetric and skew-symmetric parts one can rewrite (6) as

$$\underline{\underline{\epsilon}} = \epsilon_a \underline{\underline{I}} + \frac{\epsilon_b}{2}(\boldsymbol{u}_m\, \boldsymbol{u}_n + \boldsymbol{u}_n\, \boldsymbol{u}_m) + \frac{\epsilon_b}{2}(\boldsymbol{u}_m\, \boldsymbol{u}_n - \boldsymbol{u}_n\, \boldsymbol{u}_m). \tag{7}$$

In (7), we recognize the first two terms on the right-hand side as having exactly the biaxial structure of (4) whereas the last term has a typical *gyrotropic* form [18]. It should be said, nevertheless, that neither the biaxial nor the gyrotropic medium can be obtained from (7) as special cases because the last two terms on the right-hand side of (7) are intricately linked†. In any case, these properties make the medium characterized by the permittivity dyadic (6) an intriguing object for detailed investigation.

## 3. The dyadic Green function

### *3.1. Solution representation*

In order to obtain a solution of (1) we use

$$\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}}) \cdot \underline{\underline{L}}_{\mathrm{adj}}(\nabla, \underline{\underline{\epsilon}}) = \underline{\underline{L}}_{\mathrm{adj}}(\nabla, \underline{\underline{\epsilon}}) \cdot \underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}}) = H_{\mathrm{det}} \underline{\underline{I}}, \tag{8}$$

a relation that serves as the definition of an *adjoint* operator $\underline{\underline{L}}_{\mathrm{adj}}(\nabla, \underline{\underline{\epsilon}})$ and a scalar operator $H_{\mathrm{det}}$. As mentioned previously, the latter is in general (i.e. for all linear mediums) a fourth-order operator, which is often called the determinant operator due to the obvious analogy to matrix algebra.

With the help of (8) a solution of (1) can formally be established in the form

$$\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}') = \underline{\underline{L}}_{\mathrm{adj}}(\nabla, \underline{\underline{\epsilon}}) G(\boldsymbol{x}, \boldsymbol{x}') \tag{9}$$

where $G(\boldsymbol{x}, \boldsymbol{x}')$ is a scalar Green function that must be calculated from

$$H_{\mathrm{det}} G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{10}$$

Indeed, (9) and (10) provide a solution representation not just for the operator $\underline{\underline{L}}(\nabla, \underline{\underline{\epsilon}})$ arising from the special type of medium considered here but for the most general, homogeneous, linear,

† In [10], page 135, it is claimed that a structure of the form (6) 'resembles (the constitutive dyadic of an) affinely uniaxial (medium)'. Yet, with its skew-symmetric component, there does not appear to be a possibility that any affine transformation can transform (6) into (3).

bianisotropic medium (of course, the operator $\underline{\underline{L}}$ and its adjoint $\underline{\underline{L}}_{\mathrm{adj}}$ will then have different specific form). It is a *formal* representation only, because (10) is in general not solvable in closed form. Equally importantly, this approach dissociates $G(\boldsymbol{x}\,,\boldsymbol{x}')$ too strongly from $\underline{\underline{L}}_{\mathrm{adj}}(\nabla,\underline{\underline{\epsilon}})$; in other words, it is possible for certain types of mediums to find a closed-form expression for $\underline{\underline{G}}(\boldsymbol{x},\boldsymbol{x}')$ even though none exists for $G(\boldsymbol{x}\,,\boldsymbol{x}')$.

Instead, we shall follow a slightly different avenue: upon substitution of (6) and usage of the identity $\nabla\times\nabla\times\underline{\underline{I}} = \nabla\nabla - \nabla^2\underline{\underline{I}}$ ($\nabla^2$ is the Laplace operator), (2) becomes

$$\underline{\underline{L}}(\nabla,\underline{\underline{\epsilon}}) = -(\nabla^2 + k^2)\underline{\underline{I}} + \nabla\nabla - k^2\tau\boldsymbol{u}_m\,\boldsymbol{u}_n \tag{11}$$

where the abbreviations $k^2 = \omega^2\epsilon_a\mu$ and $\tau = \epsilon_b/\epsilon_a$ have been used. Standard methods now permit the extraction of $\underline{\underline{L}}_{\mathrm{adj}}(\nabla,\underline{\underline{\epsilon}})$ and $H_{\mathrm{det}}$ and we find

$$\underline{\underline{L}}_{\mathrm{adj}}(\nabla,\underline{\underline{\epsilon}}) = H_m\underline{\underline{L}}_e(\nabla,\underline{\underline{\epsilon}}) - k^2\tau(\nabla\times\boldsymbol{u}_n)(\nabla\times\boldsymbol{u}_m) \tag{12}$$

$$H_{\mathrm{det}} = -k^2(1+\tau\boldsymbol{u}_m\cdot\boldsymbol{u}_n)H_eH_m. \tag{13}$$

Therein we have introduced a dyadic operator

$$\underline{\underline{L}}_e(\nabla,\underline{\underline{\epsilon}}) = \nabla\nabla + k^2(1+\tau\boldsymbol{u}_m\cdot\boldsymbol{u}_n)\epsilon_a\underline{\underline{\epsilon}}^{-1} \tag{14}$$

where $\underline{\underline{\epsilon}}^{-1}$, the inverse dyadic of $\underline{\underline{\epsilon}}$, is calculated as

$$\underline{\underline{\epsilon}}^{-1} = \frac{1}{\epsilon_a}\left(\underline{\underline{I}} - \frac{\tau}{1+\tau\boldsymbol{u}_m\cdot\boldsymbol{u}_n}\boldsymbol{u}_m\,\boldsymbol{u}_n\right). \tag{15}$$

Also, there are the two scalar, second-order operators

$$H_e = \nabla^2 - \frac{\tau}{1+\tau\boldsymbol{u}_m\cdot\boldsymbol{u}_n}(\nabla\times\boldsymbol{u}_m)\cdot(\nabla\times\boldsymbol{u}_n) + k^2 \tag{16}$$

$$H_m = \nabla^2 + k^2. \tag{17}$$

We note that $H_m$ is a standard, isotropic Helmholtz operator due to the magnetic isotropy of the medium whereas $H_e$ is a Helmholtz-like operator. It is also apparent from (13) that, as anticipated for the medium under consideration and in accordance with (5), $H_{\mathrm{det}}$ does indeed factorize into a product of two second-order operators.

Further manipulation then leads to the complete representation of $\underline{\underline{G}}(\boldsymbol{x},\boldsymbol{x}')$:

$$\underline{\underline{G}}(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{k^2(1+\tau\boldsymbol{u}_m\cdot\boldsymbol{u}_n)}[\underline{\underline{L}}_e(\nabla,\underline{\underline{\epsilon}})g_e(\boldsymbol{x}\,,\boldsymbol{x}') + k^2\tau\underline{\underline{M}}(\boldsymbol{x},\boldsymbol{x}')]. \tag{18}$$

Here we have introduced a scalar Green function $g_e(\boldsymbol{x}\,,\boldsymbol{x}')$ that satisfies

$$H_e g_e(\boldsymbol{x}\,,\boldsymbol{x}') = -\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{19}$$

and a dyadic function $\underline{\underline{M}}(\boldsymbol{x},\boldsymbol{x}')$:

$$H_eH_m\underline{\underline{M}}(\boldsymbol{x},\boldsymbol{x}') = (\nabla\times\boldsymbol{u}_n)(\nabla\times\boldsymbol{u}_m)\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{20}$$

The following two sections will address the problem of finding solutions to (19) and (20).

### 3.2. *The scalar Green function*

To extract a solution of (19), we choose, without loss of generality [5], the following representation for $\boldsymbol{u}_m$ and $\boldsymbol{u}_n$:

$$\boldsymbol{u}_m = \boldsymbol{u}_x \qquad \boldsymbol{u}_n = \boldsymbol{u}_x\cos\phi + \boldsymbol{u}_y\sin\phi \tag{21}$$

with the consequence $\boldsymbol{u}_m \cdot \boldsymbol{u}_n = \cos\phi$, while $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$ is the triplet of Cartesian unit vectors. Denoting $\eta = \tau/(1 + \tau\cos\phi)$ and indicating partial derivatives with subscripts, we then have

$$H_e = \partial_{xx} + (1 - \eta\cos\phi)(\partial_{yy} + \partial_{zz}) + \eta\sin\phi\,\partial_{xy} + k^2. \tag{22}$$

The mixed derivative $\partial_{xy}$ may be eliminated by the orthogonal coordinate transformation

$$\overline{\boldsymbol{x}} = \underline{\underline{T}} \cdot \boldsymbol{x} = \begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) & 0 \\ -\sin(\phi/2) & \cos(\phi/2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \boldsymbol{x} \tag{23}$$

amounting to a rotation of the coordinate system by an angle of $\phi/2$ in the $xy$ plane (bisecting $\boldsymbol{u}_m$ and $\boldsymbol{u}_n$). Consequently, (19) is transformed into

$$(a_x\partial_{\overline{xx}} + a_y\partial_{\overline{yy}} + a_z\partial_{\overline{zz}} + k^2)g_e(\overline{\boldsymbol{x}}\,, \overline{\boldsymbol{x}}') = -\delta(\overline{\boldsymbol{x}} - \overline{\boldsymbol{x}}') \tag{24}$$

with

$$\begin{aligned} a_x &= 1 - \eta\cos\phi\sin^2(\phi/2) + (\eta/2)\sin^2\phi \\ a_y &= 1 - \eta\cos\phi\cos^2(\phi/2) - (\eta/2)\sin^2\phi \\ a_z &= 1 - \eta\cos\phi. \end{aligned} \tag{25}$$

The scalar operator in the differential equation (24) is now just a scaled Helmholtz operator and a solution can be derived without complications. Upon transformation back to the original variables, the scalar Green function $g_e(\boldsymbol{x}\,, \boldsymbol{x}')$ can thus be given as

$$g_e(\boldsymbol{x}\,, \boldsymbol{x}') = \frac{1}{\sqrt{a_x a_y a_z}} \frac{\exp[\mathrm{i}kD(\boldsymbol{x}, \boldsymbol{x}')]}{4\pi D(\boldsymbol{x}, \boldsymbol{x}')} \tag{26}$$

where $D(\boldsymbol{x}, \boldsymbol{x}')$ is a modified distance function:

$$D^2(\boldsymbol{x}, \boldsymbol{x}') = \underline{\underline{T}} \cdot (\boldsymbol{x} - \boldsymbol{x}') \cdot \underline{\underline{A}}^{-1} \cdot \underline{\underline{T}} \cdot (\boldsymbol{x} - \boldsymbol{x}') \tag{27}$$

with $\underline{\underline{A}} = \mathrm{diag}(a_x, a_y, a_z)$.

### 3.3. The dyadic function

There are two approaches to solving a dyadic fourth-order equation of the form (20) with a product of two second-order scalar operators acting on the unknown function. In previous work dealing with such differential equations, a spatial Fourier transform (in all three coordinates) was employed and the arising integrals could then be explicitly evaluated by using cylindrical coordinates in Fourier space [2] (see also [19]). Alternatively, a more direct method was shown to lead to the same results, also using cylindrical coordinates—but in $\boldsymbol{x}$ space and without making recourse to a Fourier transform [20].

However, where these procedures worked for uniaxial dielectric [2], as well as uniaxial dielectric–magnetic [19] and certain classes of uniaxial bianisotropic mediums [21, 20, 16], they fail here because the medium does not have rotational symmetry with respect to a specific axis (in other words, cylindrical symmetry). That lack of rotational symmetry is easily apparent from (20) both in the scalar operator $H_e$ on its left-hand side and in the dyadic operator acting on the delta function on the right-hand side. If and only if $\boldsymbol{u}_m = \boldsymbol{u}_n$, i.e. one specializes to uniaxiality, is cylindrical symmetry restored.

Equally, using yet again the orthogonal coordinate transformation (23) only manages to transform $H_e$ into the form (22) while the (isotropic) Helmholtz operator $H_m$ remains invariant. Further transformation of $H_e$ into a form that does have cylindrical symmetry with respect to one axis will modify $H_m$ also into an operator with cylindrical symmetry—but the two axes to which the symmetries of the transformed versions of $H_e$ and $H_m$ refer will not coincide.

As a consequence, there does not appear to be a possibility of extracting a closed-form solution for $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ from (20). The best that can be achieved is a representation of $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ through three spatial Fourier integrals. One of the integrations can be performed by using the method of residues. However, as such a representation does not qualify as a closed-form solution in the spirit of our defined requirement above, we refrain from pursuing such an approach.

It is however instructive to see the connection between $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ and the scalar Green functions $g_e(\boldsymbol{x}, \boldsymbol{x}')$, given by (26), and its counterpart $g_m(\boldsymbol{x}, \boldsymbol{x}')$:

$$g_m(\boldsymbol{x}, \boldsymbol{x}') = \frac{\exp(\mathrm{i}k|\boldsymbol{x} - \boldsymbol{x}'|)}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} \tag{28}$$

which is a solution of $H_m g_m(\boldsymbol{x}, \boldsymbol{x}') = -\delta(\boldsymbol{x} - \boldsymbol{x}')$. Decomposition of the fourth-order operator $H_e H_m$ leads to

$$[(\nabla \times \boldsymbol{u}_m) \cdot (\nabla \times \boldsymbol{u}_n)]\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{\eta}(\nabla \times \boldsymbol{u}_n)(\nabla \times \boldsymbol{u}_m)[g_m(\boldsymbol{x}, \boldsymbol{x}') - g_e(\boldsymbol{x}, \boldsymbol{x}')]. \tag{29}$$

In expanded form, the partial differential operator acting on $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ in (29) is: $-\sin\phi\,\partial_{xy} + \cos\phi\,(\partial_{yy} + \partial_{zz})$.

Even though $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ cannot be explicitly found, some conclusions can be drawn for the electric field itself. It follows from (20) that

$$H_e H_m \boldsymbol{u}_n \cdot \underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{0} \qquad H_e H_m \underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{u}_m = \boldsymbol{0}. \tag{30}$$

For an infinite-medium solution the complementary function (of the defining differential equation) is irrelevant and it follows thus that

$$\boldsymbol{u}_n \cdot \underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{0} \qquad \underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{u}_m = \boldsymbol{0}. \tag{31}$$

The electric field $\boldsymbol{E}(\boldsymbol{x})$ is represented by

$$\boldsymbol{E}(\boldsymbol{x}) = \mathrm{i}\omega\mu \int_{x'} \underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{J}(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}' \tag{32}$$

where $\boldsymbol{J}$ is the electric current density. It can therefore be seen that, for the component of $\boldsymbol{E}$ parallel to $\boldsymbol{u}_n$, $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ is fully expressed through the first term in the square bracket of (18) and $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ is of no relevance for that field component. The same holds for fields generated by the component of $\boldsymbol{J}$ parallel to $\boldsymbol{u}_m$.

The fact that the dyadic function $\underline{\underline{M}}(\boldsymbol{x}, \boldsymbol{x}')$ is related to the scalar Green functions $g_e(\boldsymbol{x}, \boldsymbol{x}')$ and $g_m(\boldsymbol{x}, \boldsymbol{x}')$ through a differential equation (29) follows on from the scalar properties of factorizable fourth-order operators. Consider a scalar Green function $W(\boldsymbol{x}, \boldsymbol{x}')$ defined by

$$H^{(4)} W(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{33}$$

where the fourth-order scalar partial differential operator $H^{(4)}$ can be factorized into a product as per

$$H^{(4)} = H_1 H_2. \tag{34}$$

Therein, $H_1$ and $H_2$ are two Helmholtz-like operators of second order with corresponding scalar Green functions $g_1(\boldsymbol{x}, \boldsymbol{x}')$ and $g_2(\boldsymbol{x}, \boldsymbol{x}')$ defined by

$$H_n g_n(\boldsymbol{x}, \boldsymbol{x}') = -\delta(\boldsymbol{x} - \boldsymbol{x}') \qquad H_n = \nabla \cdot \underline{\underline{a}}_n \cdot \nabla + \lambda_n \qquad (n = 1, 2) \tag{35}$$

where $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ are constant dyadics and $\lambda_1$ and $\lambda_2$ are scalar parameters.

It follows from (33)–(35) that $g_1$ and $g_2$ are derivable from $W$ in the form

$$g_1(\boldsymbol{x}, \boldsymbol{x}') = -H_2 W(\boldsymbol{x}, \boldsymbol{x}') \qquad g_2(\boldsymbol{x}, \boldsymbol{x}') = -H_1 W(\boldsymbol{x}, \boldsymbol{x}'). \tag{36}$$

Expressing $W$ through $g_1$ and $g_2$ is not so straightforward, however. One obtains

$$(\lambda_1 \nabla \cdot \underline{\underline{a}}_2 \cdot \nabla - \lambda_2 \nabla \cdot \underline{\underline{a}}_1 \cdot \nabla) W(\boldsymbol{x}, \boldsymbol{x}') \\ = (\nabla \cdot \underline{\underline{a}}_1 \cdot \nabla) g_1(\boldsymbol{x}, \boldsymbol{x}') - (\nabla \cdot \underline{\underline{a}}_2 \cdot \nabla) g_2(\boldsymbol{x}, \boldsymbol{x}'). \tag{37}$$

The last relation is in general a differential equation, highlighting the fact that Green functions of fourth-order operators do not simply 'decay' into Green functions of second-order operators.

Only in the very restricted situation where $\nabla \cdot \underline{\underline{a}}_1 \cdot \nabla = \nabla \cdot \underline{\underline{a}}_2 \cdot \nabla$, i.e. $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ have identical symmetric parts, does (37) reduce to the simple algebraic relation $W = (g_1 - g_2)/(\lambda_1 - \lambda_2)$, assuming $\lambda_1 \neq \lambda_2$.

## 4. Conclusion

The medium considered here is characterized by two distinct axes ($\boldsymbol{u}_m$ and $\boldsymbol{u}_n$, respectively). It is thus more general than a uniaxial medium but falls short of the definition of a truly biaxial medium. Its main interest arises from its status as the most general type of anisotropic dielectric medium that, according to the sufficient condition (5), has a determinant operator that is factorizable into the product of two second-order, scalar differential operators of Helmholtz type.

The availability of such a factorization property is a key ingredient in the derivation of closed-form expressions for dyadic Green functions. The determinant operators of all linear mediums for which closed-form expressions exist to date have this property. The main results of this paper are the derivation of the dyadic Green function representation (18), the explicit calculation of $g_e(\boldsymbol{x}, \boldsymbol{x}')$ in (26) and the *conjecture* that the dyadic differential equation (20) cannot be solved explicitly (in the case $\boldsymbol{u}_m \neq \boldsymbol{u}_n$) and that thus no closed-form solution for $\underline{\underline{G}}(\boldsymbol{x}, \boldsymbol{x}')$ exists. Based on this conjecture, two conclusions emerge.

(i) While all currently known closed-form dyadic Green functions are based on factorizable determinant operators, factorization is not a sufficient condition for the existence of closed-form, infinite-medium dyadic Green functions.

(ii) Within the class of anisotropic dielectric mediums, the uniaxial dielectric medium (or any medium that can be reduced to such a medium by, for example, affine transformations) remains the most general medium for which a closed-form, infinite-medium dyadic Green function has been derived†.

A final point that is worthy of illumination contains the electromagnetic field in the source region of the medium characterized by (6). The field in the source region (or the near zone) is an important ingredient for homogenization theories of composite mediums. While some mathematical approaches to its derivation require the explicit knowledge of dyadic Green functions (see [23] for an up-to-date review), a Fourier technique is able to extract the field in the source region as was shown for the most general, linear, anisotropic medium [24]. In [25]—where the results were further generalized to bianisotropic mediums—it was first observed that in the key quantity that determines the field in the source region, the so-called

† This remark requires some clarification: closed-form dyadic Green functions have indeed been found for more general types of anisotropic mediums. Yet, the price being paid for a larger parameter space of constitutive parameters is that such formulae only become available when specific algebraic conditions between the parameters are fulfilled. These conditions are motivated purely by mathematical necessities and not based on any fundamental symmetry or other property of the medium. One such example is the *gyrotropic dielectric–magnetic* medium considered in [22]. The condition that permits a closed-form solution intricately links the permittivity and permeability parameters in a way that is not recognized by any realistic model of gyrotropy. It also means that the existence of such an algebraic condition does not permit reduction of the results to the well known gyrotropic dielectric or the gyrotropic magnetic medium, respectively. While mathematically commendable, such derivations have therefore limited physical relevance.

depolarization dyadic, any skew-symmetric parts in the constitutive dyadics are filtered out and do not contribute to the field structure. This means in the present context that the medium characterized by (6) has a source region field that is identical to that of the biaxial, dielectric medium (4). This becomes immediately apparent if the alternative version of (6), which is (7), is inspected: (4) and (7) differ only by a skew-symmetric part. Explicit formulae for the depolarization dyadic of a biaxial dielectric medium—which thus also apply in full to the medium considered here—in terms of elliptic functions can be found elsewhere [26].

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